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Lee waves in stratified flows with simple harmonic time dependence

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The process of internal gravity wave generation by the simple harmonic flow ($U = U_0 \cos \omega_0 t$) of a stably stratified fluid (Brunt–Väisälä frequency N) over an obstacle is investigated in some detail. Attention is primarily directed to the behaviour of the solution in various limiting cases, and to estimating the flux of energy into the internal wave field. In general, waves are generated not only at the fundamental frequency ω_0, but also at all of its harmonics. But, for values of ω_0/N greater than about one half, the waves of fundamental frequency are dominant. For values of ω_0/N less than about one half, the quasi-static approximation, in which the problem is considered as a slowly-varying version of the classical lee wave problem, is found to provide a viable estimate for the wave field. The general solution is found to compare favourably with the limited available experimental data.

1. Introduction

The generation of internal gravity waves by the flow of a density stratified fluid over an obstacle is a problem of great geophysical interest, which has attracted considerable attention over the years (see e.g. Miles 1969; Zeytounian 1969*a*, *b*). Current interest in the problem derives in large measure from the fact that the momentum and energy transported by such waves may have a significant effect on large-scale geophysical flows (Lilly 1972). For the most part, work on this problem has been restricted to the meteorological context, in which the basic flow field may be considered steady. The corresponding oceanographic problem is somewhat complicated by the fact that, in addition to any steady background flows, we must also contend with significant time-dependent components of the flow associated with the ubiquitous tides. It is this latter problem to which the present work is addressed.

In considering the generation of internal waves by the back-and-forth motion of a stratified fluid over an obstacle, two important limiting cases may arise. In the first case, the time scale associated with the background motion is very large compared with the characteristic time scale associated with the buoyancy restoring force in the stratified environment, and the problem is reduced to a slowly-varying or quasi-static version of the classical steady lee wave problem, with which the vast majority of the lee wave literature is concerned. Cartwright (1959) has discussed internal wave generation by tidal flows over sea floor topography in this limit. The other limiting case, referred to here as the acoustic limit, arises when the horizontal excursion of fluid elements convected by the background flow is small compared with the horizontal length scale that characterizes the obstacle. Several authors have considered various aspects of the acoustic-limit problem, notably Görtler (1943), Mowbray & Rarity (1967), Lighthill (1967) and Hurley (1969).† The generation of internal waves in the ocean by tidal flows over bottom topography has been discussed in this limit by Cox & Sandstrom (1962) and Baines (1973). Of interest also in this regard is the work of Rattray *et al.* (1969) on the problem of the generation of internal tides at the continental margin.

By comparison with these limiting cases, the general problem has received very little attention. Apart from the author's doctoral dissertation (Bell 1973), parts of which serve as a basis for the present study, the only relevant work appears to be that of Mork (1968), who considered the problem of internal wave generation by simple harmonic flow over an element of bottom topography in a stratified fluid of limited vertical extent, using the long-wave or hydrostatic approximation in the wave field. Mork's solution for the resultant wave pattern is represented by a summation of non-dispersive internal wave modes, and is restricted in its validity to situations in which the relevant horizontal length scales of the problem are significantly greater than the depth of the fluid. There is no discussion of important second-order properties of the wave field, such as the flux of energy away from the obstacle which forces the waves.

The present study attempts to elucidate the process of internal wave generation in the general case of the simple harmonic flow of a stratified fluid over an obstacle. For the sake of clarity, we reduce the problem to its essential elements of stable density stratification and a background flow with simple harmonic time dependence. We adopt a somewhat different approach to the problem from that taken by Mork (1968), in that we consider a fluid of unlimited vertical extent. Attention is thus directed to the generation process, uncluttered by any aspects of the propagation of internal gravity waves which may distract our attention. In §2 we develop the fundamental linearized solution for the internal wave field generated by the flow of a uniformly stratified fluid with basic velocity

$$U = U_0 \cos \omega_0 t$$

over a two-dimensional obstacle. The properties of this fundamental solution are discussed in some detail in §3, including the passage of the solution to the quasi-static and acoustic limits. In the general case, waves are generated not only at the fundamental frequency ω_0, but also at all of its harmonics; the fundamental frequency waves are sensitive to the slope of the obstacle, those of frequency $2\omega_0$ to its curvature, and so forth for the higher harmonics. Only those waves of frequency less than the ambient Brunt–Väisälä frequency N are freely propagating, so that, in the far field, the wave pattern resembles a superposition of n_0 V-shaped patterns when viewed from a reference frame convected by the basic flow, where n_0 is the largest integer less than N/ω_0. In §4 the solution is

† The discussion in Greenspan (1968, §4.4), of the analogous acoustic-limit problem in a rotating fluid, is also of interest.

found to compare favourably with the limited available experimental evidence (Lee 1972). Finally, § 5 is devoted to the consideration of the flux of energy away from the obstacle. An expression is derived for the time average power in the internal wave field, i.e. for the rate at which energy from the basic flow is converted into internal wave energy. The behaviour of the time average power as a function of the sensible parameters of the problem is investigated by considering as an example the flow over an obstacle of shape $1/(1+x^2)$, the classical Witch of Agnesi. It is found that, for values of β (defined as $\sqrt{2}NL/U_0$, where L is the characteristic width of the obstacle) somewhat greater than unity, the acoustic-limit approximation provides a reasonable estimate of the power in the wave field. Somewhat surprisingly, it is also found that, for values of ω_0/N less than about one half, the quasi-static approximation provides a not unreasonable estimate of the power for all β. Bell (1973) discusses the geophysical implications of this problem.

2. Fundamental solution

We consider the flow of a stably stratified fluid over an obstacle for the case in which the forcing exhibits a simple harmonic time dependence. To render the problem tractable, we assume that the obstacle is such that its presence produces only a small perturbation to the flow field that would exist in its absence. The presence of the obstacle then presents a localized disturbance to the system; and it is a general property of stably stratified systems that disturbance energy is propagated away from the source region in the form of an internal gravity wave field. Now, in this study, we are interested in the generation of these waves. The essential elements that define the problem are stable density stratification and a basic unperturbed flow structure which varies harmonically with time. So we consider the uncluttered two-dimensional problem of the flow of an unbounded uniformly stratified ($N^2 = \text{const.}$, where N is the Brunt–Väisälä frequency in terms of the acceleration of gravity and the density; $N^2 = -(g/\rho)\,d\rho/dz$)) Boussinesq fluid, over an obstacle, when the unperturbed flow structure is simply a spatially uniform horizontal flow with simple harmonic time dependence ($U = U_0 \cos \omega_0 t$), as illustrated in figure 1.

Although Coriolis forces may be dynamically significant in oceanographic applications, our model is considered in a non-rotating reference frame. The inclusion of the effects of rotation is rather straightforward (see Bell 1973), and does not substantially alter the physical picture that emerges from the model considered here. Our consideration of a spatially uniform semi-infinite environment should not be construed as implying that in specific applications such effects as spatial inhomogeneity of the background state or the presence of boundaries at which internal wave energy is reflected will not be important. Indeed, they may be. Rather, such effects more properly belong not to the study of internal wave generation, but to the study of internal wave propagation. Here, we are concerned with the generation problem, and it is felt that the model considered here captures the essential physics of the problem. The wave generation process should be adequately described by the model in those situations

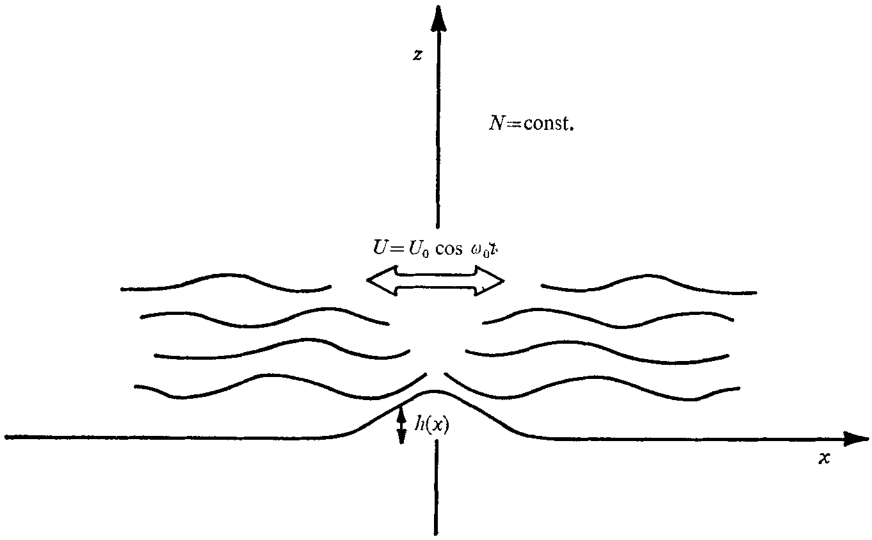

FIGURE 1. Definition sketch illustrating model of back-and-forth flow of a stably stratified fluid over an obstacle.

where the length scales characterizing variations in the background state (including the distances to boundaries) are substantially larger than the relevant length scales which characterize the model (U_0/N and geometric scales associated with the obstacle). The reader interested in questions of propagation is referred to Bretherton (1971) and Budden (1961).

Small perturbations to the equilibrium state described above are governed by the equations

$$u_t + Uu_x = -\frac{1}{\rho_0} p_x, \tag{2.1a}$$

$$w_t + Uw_x = -\frac{1}{\rho_0} p_z - \frac{g}{\rho_0}\rho, \tag{2.1b}$$

$$\rho_t + U\rho_x = \frac{\rho_0}{g} N^2 w, \tag{2.1c}$$

$$u_x + w_z = 0. \tag{2.1d}$$

ρ, p, u, w are respectively the perturbation density, pressure and horizontal and vertical velocity components; ρ_0 is a reference density; and independent variables in subscript position denote partial differentiation. Equations (2.1 a–d) are readily transformed into a single equation for the perturbation vertical velocity

$$D^2\nabla^2 w + N^2 w_{xx} = 0. \tag{2.2}$$

The operator D is defined by

$$D = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}, \tag{2.3}$$

and ∇^2 is the Laplace operator. The operator D is not autonomous; rather it depends explicitly on time t through the time-dependent velocity field $U(t)$.

Boundary conditions in x and t for the solution of (2.2) are provided by the requirements that w be Fourier transformable in x and periodic in t. The boundary conditions in z provide the essence of the solution. At the bottom ($z = 0$), the linearized boundary condition is that

$$w = U\,dh/dx, \tag{2.4}$$

where $h(x)$ is the height of the obstacle which forces the motion. Within the linear approximation, this is equivalent to the condition that the total flow field (background plus waves) be locally tangential to the bottom topography. Above the bottom, we apply a radiation condition of upward energy flux, consistent with our notion of causality.

The familiar linearized bottom boundary condition (2.4) in essence results from the assumption that the slopes of those wave rays that characterize the wave pattern be significantly greater than the characteristic slope H/L of the obstacle that forces the motion, where H and L are respectively the characteristic height and width of the obstacle. In considering this restriction imposed by the linearization, two important cases may arise, depending on the magnitudes of the ratios ω_0/N and $U_0/\omega_0 L$, where U_0/ω_0 is identifiable as the amplitude of the horizontal motion in the basic state. The limit $\omega_0/N \to 0$ is recognized as a quasi-static limit in which the time derivatives in the governing equation may be consistently ignored, with the problem reducing to a slowly-varying version of the classical problem of lee waves in a steady stratified flow. In this case, U/L becomes the characteristic frequency of the waves, and the representative slope of the wave rays is then U/NL, so that, when $\omega_0/N \ll 1$, we require that $NH/U \ll 1$.

In the other important limiting case, corresponding to $U_0/\omega_0 L \to 0$, the problem reduces essentially to the other classical problem of a vibrating disturbance in a stationary stratified fluid, referred to here as the acoustic limit. In such a case, the representative slope of the wave rays is ω_0/N, so that when $U_0/\omega_0 L \ll 1$ we require that $NH/\omega_0 L \ll 1$. In the general case where the ratios ω_0/N and $U_0/\omega_0 L$ are of the order unity, the conditions are equivalent. If the appropriate linearization condition is violated, nonlinear upstream influence or blocking effects may prevail. The nature of such nonlinearities is discussed by Miles (1971); of interest also are the experiments of Browand & Winant (1972). Of course, validity of the linearization in the boundary condition ensures validity of the linearization in the governing equations.

Returning to the solution of the problem, we introduce the Fourier transform

$$(\hat{\ }) = \int_{-\infty}^{\infty} (\ )\exp(-i\kappa x)\,dx, \tag{2.5a}$$

with inverse

$$(\ ) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{\ })\exp(i\kappa x)\,d\kappa, \tag{2.5b}$$

and the governing equation (2.2) becomes

$$\hat{D}^2\hat{w}_{zz} - \kappa^2(N^2+\hat{D}^2)\,\hat{w} = 0, \tag{2.6}$$

where $\hat{D} = \partial/\partial t + i\kappa U$.

It is convenient to introduce a co-ordinate system moving with the basic state. To this end, we set

$$\hat{w} = \tilde{w} \exp\left(-i\kappa \int_0^t U(\tau)\, d\tau\right). \tag{2.7}$$

Then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{w} \exp(i\kappa\xi)\, d\kappa = w(x,z,t), \tag{2.8a}$$

where

$$\xi = x - \int_0^t U(\tau)\, d\tau \tag{2.8b}$$

is the horizontal co-ordinate in a reference frame fixed with respect to the background flow. The bottom boundary condition is then transformed to

$$\tilde{w}(\kappa,0,t) = \hat{h}\frac{\partial}{\partial t}\exp\left(i\kappa\int_0^t U(\tau)\, d\tau\right),$$

which, with $U = U_0 \cos \omega_0 t$, may be expressed as

$$\tilde{w}(\kappa,0,t) = \hat{h} \sum_{n=-\infty}^{\infty} (in\omega_0)\exp(in\omega_0 t)\, J_n(\kappa U_0/\omega_0), \tag{2.9}$$

where J is the Bessel function of the first kind (see Watson 1966, §2.22).

We now may seek a Fourier series solution of the form

$$\tilde{w}(\kappa,z,t) = \hat{h}(\kappa) \sum_{n=-\infty}^{\infty} \tilde{W}_n(\kappa;z)\exp(in\omega_0 t)\, J_n(\kappa U_0/\omega_0),$$

where the amplitude function $\tilde{W}_n$ must satisfy

$$\tilde{W}_n'' + \kappa^2\left\{\frac{N^2}{n^2\omega_0^2} - 1\right\}\tilde{W}_n = 0, \tag{2.10}$$

with primes denoting differentiation with respect to z. The solution of (2.10), subject to the bottom boundary condition $\tilde{W}_n(\kappa;0) = in\omega_0$, is

$$\tilde{W}_n(\kappa;z) = in\omega_0 \exp(i\mu_n z), \tag{2.11}$$

where

$$\mu_n^2 = \kappa^2\left\{\frac{N^2}{n^2\omega_0^2} - 1\right\};$$

and the solution must be rendered determinate by the proper choice of $\operatorname{sgn}\mu_n$, as follows. The solution is composed of a spectrum of component waves of the form $A_n \exp i\phi_n$, where the phase function ϕ_n is such that

$$\mu_n^2 = \left(\frac{\partial\phi_n}{\partial z}\right)^2 = \kappa^2\left\{\frac{N^2}{n^2\omega_0^2} - 1\right\},$$

$$\omega_n = -\frac{\partial\phi_n}{\partial t} = -n\omega_0.$$

In the oscillatory case ($N^2 > n^2\omega_0^2$), the vertical component of the group velocity is then

$$\frac{\partial\omega_n}{\partial\mu_n} = \frac{\mu_n}{n\omega_0}\frac{n^2\omega_0^2}{\kappa^2+\mu_n^2},$$

which is positive, corresponding to upward energy propagation, for

$$\operatorname{sgn}\mu_n = \operatorname{sgn} n.$$

In the evanescent case ($N^2 < n^2\omega_0^2$), we must choose the solution that decays with height. In general, then, we have, for $N^2 > n^2\omega_0^2$,

$$\mu_n = \frac{|\kappa|}{n\omega_0}(N^2 - n^2\omega_0^2)^{\frac{1}{2}}, \tag{2.12a}$$

and, for $N^2 < n^2\omega_0^2$,

$$\mu_n = i\left|\frac{\kappa}{n\omega_0}\right|(n^2\omega_0^2 - N^2)^{\frac{1}{2}}. \tag{2.12b}$$

The general solution is then given by

$$w(x,z,t) = \frac{i}{2\pi}\sum_{n=-\infty}^{\infty} n\omega_0 \int_{-\infty}^{\infty} \hat{h}(\kappa)\, J_n(\kappa U_0/\omega_0)\exp i[(\kappa\xi + \mu_n z + n\omega_0 t)]\, d\kappa. \tag{2.13}$$

Since $h(x)$ is real, we may express the general solution in the form

$$w(x,z,t) = -\sum_{n=1}^{\infty}\frac{n\omega_0}{\pi}\operatorname{Im}\int_{-\infty}^{\infty}\hat{h}(\kappa)\, J_n(\kappa U_0/\omega_0)\exp[i(\kappa\xi + \mu_n z + n\omega_0 t)]\, d\kappa, \tag{2.14}$$

where ξ, given by (2.8*b*), is the horizontal position in a reference frame moving with the background flow.

3. Properties of the solution

The amplitude η of the wave field is defined by the linearized equation

$$w = \eta_t + U\eta_x,$$

so that, referring to (2.14), we may express the amplitude of the disturbance as

$$\eta(x,z,t) = \frac{1}{\pi}\sum_{n=1}^{\infty}\operatorname{Re}\int_{-\infty}^{\infty}\hat{h}(\kappa)\, J_n\left(\frac{\kappa U_0}{\omega_0}\right)\exp[i(\kappa\xi + \mu_n z + n\omega_0 t)]\, d\kappa, \tag{3.1}$$

where μ_n is given by (2.12), and ξ is horizontal distance measured in a reference frame moving with the basic flow, as given by equation (2.8*b*). For z = const., the integral in (3.1) is simply proportional to $\exp in\omega_0 t$, provided that the coordinate x is varied with time in such a way as to duplicate the motion of a fluid element convected by the basic flow. This solution is a superposition of wave motions not only at the fundamental frequency ω_0, but also at all of its harmonics. But only those contributions such that $n < N/\omega_0$ are correctly identified as wavelike, the contributions for $n > N/\omega_0$ being vertically evanescent (i.e. exhibiting an exponential decay with height above the bottom). In the far field, then, for ξ and z large, we need only consider the oscillating contribution to the solution. Designating the oscillatory or wavelike part of the solution by η_w, we then have

$$\eta_w(x,z,t) = \frac{1}{\pi}\sum_{n=1}^{n_0}\operatorname{Re}\int_{-\infty}^{\infty}\hat{h}(\kappa)\, J_n\left(\frac{\kappa U_0}{\omega_0}\right)\exp[i(\kappa\xi + \mu_n z + n\omega_0 t)]\, d\kappa, \tag{3.2}$$

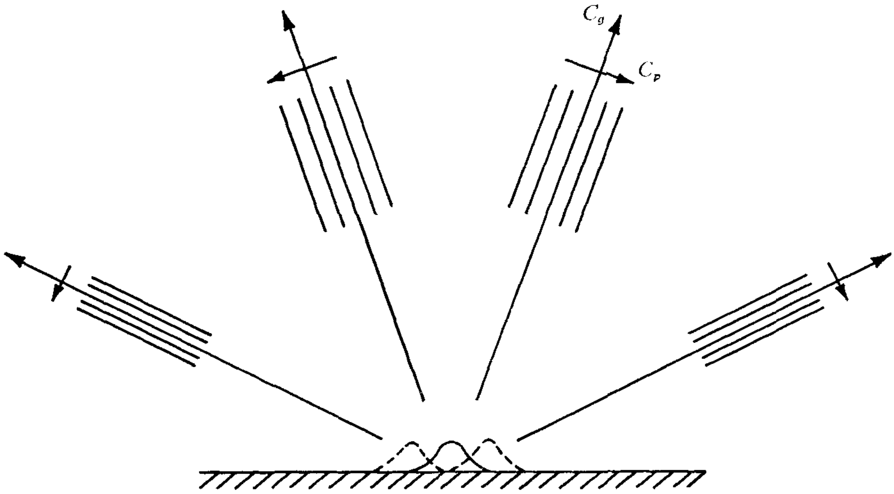

FIGURE 2. Schematic configuration of wave field for $\omega_0 = 0{\cdot}45N$, viewed in reference frame fixed with respect to background flow. – – –, maximum excursion of obstacle; long arrows, directions of energy propagation (C_g); short arrows, directions of phase propagation (C_p).

where n_0 is the largest integer less than N/ω_0. For the purposes of discussing the far-field behaviour of the solution, it is more convenient to express the oscillatory contribution as

$$\eta_w(x,z,t) = \frac{1}{\pi}\sum_{n=1}^{n_0} \operatorname{Re}\int_0^\infty \hat{h}(\kappa)\, J_n\left(\frac{\kappa U_0}{\omega_0}\right) \exp\left[i(\kappa\xi + \mu_n z + n\omega_0 t)\right] d\kappa$$
$$+\frac{1}{\pi}\sum_{n=1}^{n_0} (-1)^n \operatorname{Re}\int_0^\infty \hat{h}(\kappa)\, J_n\left(\frac{\kappa U_0}{\omega_0}\right) \exp\left[i(\kappa\xi - \mu_n z - n\omega_0 t)\right] d\kappa. \quad (3.3)$$

Invoking a classical argument, we expect that, as $\xi, z \to \infty$, the rapid oscillation of the complex exponentials will nullify any sensible contribution to the wave field except along those lines such that $z = \pm R_n \xi$, where

$$R_n = \frac{n\omega_0}{(N^2 - n^2\omega_0^2)^{\frac{1}{2}}}.$$

These lines are readily identified as the characteristics of internal waves of frequency $n\omega_0$ (i.e. those lines along which the internal wave energy propagates away from the disturbance). In a reference frame fixed with respect to the background flow (i.e. one in which the obstacle moves harmonically back and forth along the bottom), the wave pattern should then resemble a superposition of n_0 V-shaped patterns with vertices centred at $(\xi, z) = (0, 0)$, as illustrated in figure 2. Lines of constant phase are parallel to the characteristics and wave phase propagates normal to the characteristics. From the form of the phase functions $\phi_n = (\kappa\xi \pm \mu_n z \pm n\omega_0 t)$, it is apparent that, along $z = R_n \xi$, phase propagates in the direction of increasing ξ and decreasing z, while along $z = -R_n \xi$, phase propagates in the direction of decreasing ξ and decreasing z.

From the form of the solution in (3.2), it is apparent that the different harmonics are sensitive to different features of the obstacle that forces the motion. Rewriting (3.2) in the form

$$\eta_w(x,z,t) = \frac{1}{\pi}\sum_{n=1}^{n_0} \mathrm{Re}\int_{-\infty}^{\infty} [\kappa^n \hat{h}(\kappa)]\left[\kappa^{-n}J_n\left(\frac{\kappa U_0}{\omega_0}\right)\right]\exp[i(\kappa\xi+\mu_n z+n\omega_0 t)]\,d\kappa,$$

we may identify $\kappa^n\hat{h}(\kappa)$ as i^{-n} times the Fourier transform of d^nh/dx^n, provided that h is sufficiently smooth that the Fourier transform exists. We may also identify $\kappa^{-n}J_n(\kappa U_0/\omega_0)$ as the Fourier transform of a function $f_n(x)$, where (Erdelyi *et al.* 1954, §1.12)

$$f_n(x) = \begin{cases} \dfrac{1}{\pi^{\frac{1}{2}}\Gamma(n+\frac{1}{2})}\left(\dfrac{\omega_0}{2U_0}\right)^n\left(\dfrac{U_0^2}{\omega_0^2}-x^2\right)^{n-\frac{1}{2}} & (x^2 < U_0^2/\omega_0^2), \\ 0 & (x^2 > U_0^2/\omega_0^2). \end{cases} \tag{3.4}$$

Thus, provided that d^nh/dx^n is Fourier transformable for all $n \leqslant n_0$, we may represent the wave field as

$$\eta_w(x,z,t) = \frac{1}{\pi}\sum_{n=1}^{n_0} \mathrm{Re}\int_{-\infty}^{\infty} \hat{F}_n(\kappa)\exp[i(\kappa\xi+\mu_n z+n\omega_0 t-\tfrac{1}{2}n\pi)]\,d\kappa, \tag{3.5}$$

where, by the convolution theorem, $\hat{F}_n(\kappa)$ is the Fourier transform of a smoothed version of d^nh/dx^n, given by

$$F_n(x) = \int_{-\infty}^{\infty} f_n(x-x_1)\frac{d^nh}{dx_1^n}\,dx_1, \tag{3.6}$$

with $f_n(x)$ given by (3.4). Thus, the waves of fundamental frequency ω_0 are sensitive to the slope of the obstacle, the first harmonic ($n = 2$) to its curvature, and so forth for the higher harmonics. The reason for this becomes apparent if we think of the solution in terms of a formal expansion scheme, $\mathbf{u} = \mathbf{u}_1+\mathbf{u}_2+\ldots$, in which the convective acceleration term $U\partial\mathbf{u}_n/\partial x$ is systematically neglected in the governing equations for the nth contribution, but appears in the equations for the $(n+1)$th contribution as a non-homogeneous forcing term. The first term ($n = 1$) is forced by the flow over the bottom (Udh/dx), and therefore involves waves of fundamental frequency and is sensitive to the bottom slope. The second term ($n = 2$) is forced by the convective term $U\partial\mathbf{u}_1/\partial x$, and therefore involves waves of frequency $2\omega_0$ and is sensitive to the slope of the $n-1$ term (i.e. to the curvature of the bottom). The $n = 3$ term involves waves of frequency $3\omega_0$, and is sensitive to the slope of the $n = 2$ contribution, etc.

The smoothing expressed by (3.6) acts over a length $2L_0$, where, with

$$L_0 = U_0/\omega_0,$$

$2L_0$ is simply the total horizontal excursion of a fluid element convected by the background flow. At a given point in ξ, z space, then, that component of the wave field of frequency $n\omega_0$ 'sees' only a strip of bottom of length $2L_0$, centred at the point where the appropriate wave characteristic passing through the observation point intersects the bottom. In other words, for such a reference frame (in which the obstacle moves back and forth), waves of frequency $n\omega_0$ at a given observation point are influenced by only that part of the bottom topography intersecting the base of the wave characteristic appropriate to the frequency $n\omega_0$ passing through the observation point. As $L_0/L \to 0$, where L is the length scale that characterizes variations in $h(x)$, the impulse response of the filter as expressed by (3.4) behaves more and more like a Dirac spike. This is the classical acoustic limit in which fluid elements convected by the background flow move back and forth over only a small fraction of the width of the obstacle, or conversely the obstacle moves back and forth over a distance small compared with its width. In the limit then, as $\omega_0 L/U_0 \to \infty$, the wave field is given by

$$\eta_w \sim \frac{1}{\pi}\sum_{n=1}^{n_0}\frac{1}{n!}\left(\frac{U_0}{2\omega_0}\right)^n \mathrm{Re}\int_{-\infty}^{\infty} \kappa^n \hat{h}(\kappa)\exp\left[i(\kappa\xi+\mu_n z+n\omega_0 t)\right]d\kappa, \tag{3.7}$$

provided, of course, that $d^n h/dx^n$ is Fourier transformable for all $n \leqslant n_0$. From the form of (3.7), it is apparent that the fundamental frequency waves ($n = 1$) must dominate the wave field generated by flow over a smooth obstacle in the limit $\omega_0 L/U_0 \to \infty$.

The limit $\omega_0/N \to 0$ is identifiable as a quasi-static limit, in which the effect of the time derivatives in the governing equations (2.1) should be negligible. In discussing this limit, it is convenient to refer to the general solution given by (2.13), which may be expressed in terms of the wave amplitude by

$$\eta = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\eta}_n \exp(i\kappa x)\,d\kappa, \tag{3.8a}$$

where
$$\hat{\eta}_n = \hat{h}(\kappa)\exp(i\mu_n z)\,J_n\left(\frac{\kappa U_0}{\omega_0}\right)\exp\left\{i\left[n\omega_0 t-\kappa\int_0^t U(\tau)\,d\tau\right]\right\}. \tag{3.8b}$$

Expressing the Bessel function by its well-known integral representation, we then have that

$$\hat{\eta}_n = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0}\hat{h}\exp(i\mu_n z)\exp\left\{i\left[n\omega_0 t-\kappa\int_0^t U(\tau)\,d\tau\right]\right\} \times\exp\left\{-i\left[n\omega_0\theta-\kappa\int_0^\theta U(\tau)\,d\tau\right]\right\}d\theta. \tag{3.9}$$

If, as is consistent with the quasi-static approximation, we use

$$\int_0^t U(\tau)\,d\tau \sim Ut$$

in (3.9), we then have

$$\hat{\eta}_n \sim \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0}\hat{h}\exp(i\mu_n z)\exp\left[-i(n\omega_0-\kappa U)(\theta-t)\right]d\theta,$$

so that, for $n = \kappa U_0/\omega_0$,

$$\hat{\eta}_n \sim \hat{h}\exp(i\mu_n z),$$

while otherwise $\hat{\eta}_n \sim 0$. In the quasi-static limit ($\omega_0/N \to 0$), then, we have

$$\eta \sim \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{h}(\kappa)\exp\left[i(\mu z+\kappa x)\right]d\kappa, \tag{3.10}$$

n	Ray slope
1	0·13
2	0·27
3	0·43
4	0·62
5	0·87
6	1·29
7	2·37

TABLE 1. Slopes of characteristics for internal waves of frequency $n\omega_0$ in Lee's (1972) experiment.

where, for $N^2 > \kappa^2U^2$,

$$\mu = \operatorname{sgn}(\kappa U)\,(N^2/U^2 - \kappa^2)^{\frac{1}{2}}, \tag{3.11a}$$

and, for $N^2 < \kappa^2U^2$,

$$\mu = i(\kappa^2 - N^2/U^2)^{\frac{1}{2}}, \tag{3.11b}$$

in agreement with the classical lee wave result (see Miles 1969).

4. Comparison with experiment

The phase configuration described in §3, is of course, exactly that observed in the well-known experiments of Görtler (1943) and Mowbray & Rarity (1967), the latter of which incidentally illustrates clearly the generation of waves not only at the fundamental frequency, but also at its admissible harmonics. Unfortunately, these experiments, although elegant and qualitatively significant, do not permit a quantitative evaluation of the theory presented here. But the experiments Lee (1972) are useful in this connexion. Lee's work was designed primarily as a study of the generation and propagation of long nonlinear waves although one particular set of observations is relevant here.

In the experiment of interest here, an obstacle of approximate shape

$$h(x) = HL^2/(x^2+L^2) \tag{4.1}$$

(the classical Witch of Agnesi), with $H = 2{\cdot}3$ cm and $L = 3{\cdot}1$ cm, was towed back and forth along the bottom of a tank filled with stably stratified salt water ($N = 0{\cdot}63\ \mathrm{s}^{-1}$) to a depth of 38·4 cm. The fundamental period of the obstacle's motion was 76 s ($N/\omega_0 = 7{\cdot}6$), the amplitude of its excursion, L_0, being 27·5 cm. The amplitude of the wave field was measured by several conductivity probes located at a height of 12·6 cm above the bottom. Unfortunately, the representative slope of the obstacle ($H/L = 0{\cdot}74$) is not small compared with the slopes of the wave characteristics (given in table 1) in this experiment, so that we cannot expect the linear theory presented here to be strictly valid, especially in so far as the lower harmonics are concerned, although some agreement might be expected for the higher harmonics, especially $n = 7$.

Lee presents the results of this experiment in the form of wave forms observed at two probe locations. But the location of only one of the probes (probe A) is unambiguously specified: $\xi = 57{\cdot}5$ cm. We examined the ray paths corresponding to waves at the fundamental frequency and its harmonics, so as to

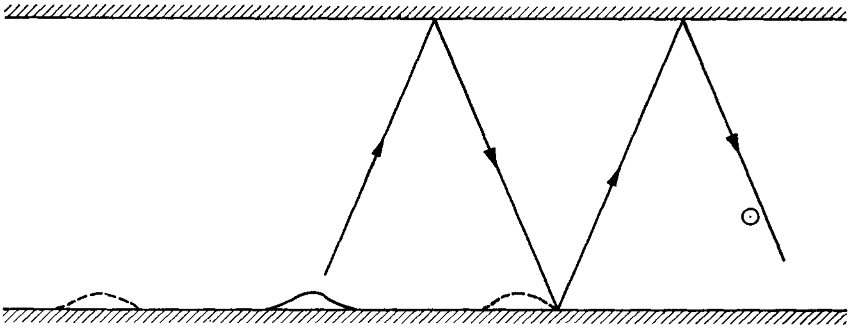

FIGURE 3. Geometry of energy propagation in Lee's (1972) experiment, drawn to scale. ➤, ray path for waves of frequency $7\omega_0$; ⊙, probe location; – – – –, maximum excursion of obstacle.

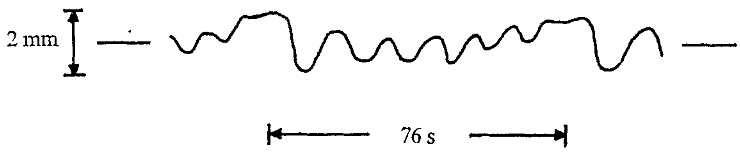

FIGURE 4. Wave form observed at probe location indicated in figure 3. Fundamental period ($2\pi/\omega_0$) is 76 s.

determine whether or not this probe lies on or near any of the characteristics. We found that the probe location corresponds to an angular deviation of approximately 0·01 rad from the ray path for the sixth harmonic ($n = 7$). The geometry is illustrated to scale in figure 3. The closest other trajectory is associated with the first harmonic ($n = 2$), with an angular deviation of some 0·06 rad from the probe location. The output of the probe is reproduced in figure 4. The wave form illustrated in this figure is obviously dominated by the sixth harmonic ($n = 7$). From (3.3), our theory predicts an amplitude of

$$|\eta_7| \sim LH \int_0^\infty \exp(-\kappa L)\, J_7(\kappa L_0)\, d\kappa,$$

where (see Watson 1966, §13.2)

$$\int_0^\infty \exp(-\kappa L)\, J_\nu(\kappa L_0)\, d\kappa = \frac{[(L^2+L_0^2)^{\frac{1}{2}} - L]^\nu}{L_0^\nu (L^2+L_0^2)^{\frac{1}{2}}}.$$

Thus, with $L = 3{\cdot}2$ cm and $L_0 = 27{\cdot}5$ cm, we predict $|\eta_7| \sim 1{\cdot}2$ mm, in good agreement with the observed wave amplitude. At the second probe, which was located at a somewhat larger but unspecified ξ co-ordinate, the signal was less coherent, although exhibiting a definite component of periodicity at a frequency of $7\omega_0$. Although rather limited, this comparison at least indicates a certain degree of consistency between the theory and experiment.

5. Energy flux

It is well known (Miles 1969) that, accompanying the flow of a stratified fluid over an obstacle, there is a systematic pressure drop from the windward to the leeward side of the obstacle, resulting in a horizontal force, the wave drag, acting on the obstacle. Within the framework of linear theory, the net horizontal force per unit cross-stream length acting on the obstacle is given by

$$F(t) = \int_{-\infty}^{\infty} p(x,0,t)\frac{dh}{dx}\,dx,$$

which, on integrating by parts and substituting from (2.1) for the pressure, and invoking Parseval's relation, may be expressed as

$$F(t) = \tfrac{1}{2}\rho_0 \frac{1}{2\pi}\int_{-\infty}^{\infty} (\hat{h}\hat{D}^*\hat{u}^* + \hat{h}^*\hat{D}\hat{u})\,d\kappa, \tag{5.1}$$

where asterisks are used to denote complex conjugates. In the problem considered here, the force as expressed by (5.1) includes not only the pressure force associated with the internal wave field, but also an acceleration reaction arising from the time-dependent nature of the background flow. Averaged over the fundamental period $2\pi/\omega_0$, the net force on the obstacle vanishes. However, the flow is doing work against the pressure force, and this rate of working is simply equal to the power in the internal wave field (i.e. the rate at which energy is being fed into the internal wave field).

Averaged over the fundamental period $2\pi/\omega_0$, the power per unit cross-stream length in the internal wave field is given by the average value of $U(t)F(t)$, i.e.

$$\overline{\mathscr{P}} = \tfrac{1}{2}\rho_0\frac{\omega_0}{4\pi^2}\int_{-\infty}^{\infty}\int_0^{2\pi/\omega_0} U(t)\,(\hat{h}\hat{D}^*\hat{u}^* + \hat{h}^*\hat{D}\hat{u})\,dt\,d\kappa. \tag{5.2}$$

Substituting from (2.1) and (2.13) and evaluating the time average as in (2.9), we obtain

$$\overline{\mathscr{P}} = \frac{2}{\pi}\rho_0 \sum_{n=1}^{n_0} n^2\omega_0^2(N^2-n^2\omega_0^2)^{\frac{1}{2}}\int_0^{\infty}\kappa^{-1}|\hat{h}|^2 J_n^2\left(\frac{\kappa U_0}{\omega_0}\right)d\kappa \tag{5.3}$$

for the time average power.

For discussion, it is convenient to normalize the power by some reference value. Miles (1969) suggested the value $\frac{1}{2}\pi\rho_0 U^2 NH^2$, where H is the height of the obstacle, as a convenient reference in dealing with steady flows. As discussed by Miles, this reference value arises from the problem of minimizing the wave drag on an obstacle of prescribed cross-sectional area and base L in the ordered limit $H/L \to 0$, $U \to 0$. To normalize the time average power, we replace U^2 by its average value $\frac{1}{2}U_0^2$, and define

$$E = \overline{\mathscr{P}}/\tfrac{1}{4}\pi\rho_0 U_0^2 NH^2. \tag{5.4}$$

Normalizing (5.3), we thus have

$$E = \frac{8}{\pi^2}\frac{1}{U_0^2 NH^2}\sum_{n=1}^{n_0} n^2\omega_0^2(N^2-n^2\omega_0^2)^{\frac{1}{2}}\int_0^{\infty}\kappa^{-1}|\hat{h}|^2 J_n^2\left(\frac{\kappa U_0}{\omega_0}\right)d\kappa. \tag{5.5}$$

If we further non-dimensionalize within (5.5), by setting

$$f = \omega_0/N, \quad k = \kappa L, \quad \zeta = \hat{h}/LH, \quad \beta = \sqrt{2}NL/U_0,$$

where L is the characteristic width of the obstacle, we obtain the convenient expression

$$E = \frac{4}{\pi^2}\beta^2 \sum_{n=1}^{n_0} n^2 f^2 (1-n^2 f^2)^{\frac{1}{2}} \int_0^\infty k^{-1} |\zeta|^2 J_n^2 \left(\frac{\sqrt{2k}}{\beta f}\right) dk, \tag{5.6}$$

for the normalized time average power in the internal wave field.

The behaviour of E as a function of the parameters f and β is perhaps best demonstrated by considering a specific example. To this end, we consider an obstacle shaped in the form of the well-known Witch of Agnesi, so that, with $h(x)$ given by (4.1),

$$\zeta(k) = \pi \exp(-|k|). \tag{5.7}$$

Using the subscript A to designate the Witch of Agnesi, we then have that

$$E_A = 4\beta^2 \sum_{n=1}^{n_0} n^2 f^2 (1-n^2 f^2)^{\frac{1}{2}} \int_0^\infty k^{-1} \exp(-2k) J_n^2 \left(\frac{\sqrt{2k}}{\beta f}\right) dk. \tag{5.8}$$

The behaviour of E_A as a function of β is illustrated in figure 5 for various values of f; the curve for $f = 0$ is based on the steady flow solution of Miles & Huppert (1969)

$$E_A|_{f=0} = \tfrac{2}{3}\beta^2[1 + \tfrac{3}{4}(\pi/\beta)\{\mathbf{L}_2(2\beta) - I_2(2\beta)\}], \tag{5.9}$$

where $\mathbf{L}$ and I are the modified Struve and Bessel functions, respectively. For $\beta \gg 1$, we may replace $J_n(\sqrt{2k}/\beta f)$ in (5.8) by its limiting form $(1/n!)\,(k/\sqrt{2}\beta f)^n$. In the acoustic limit ($\beta \to \infty$), then, the dominant contribution to the power resides in waves of fundamental frequency ($n = 1$), and

$$E_A \sim \tfrac{1}{2}(1-f^2)^{\frac{1}{2}}. \tag{5.10}$$

The asymptotic value of the power as given by (5.10) is also indicated in figure 5 for the pertinent values of f. It is apparent from figure 5 that for values of β somewhat larger than unity the acoustical-limit approximation provides a reasonable estimate of the rate at which the energy is supplied to the wave field. Somewhat surprisingly, figure 5 also indicates that, even for large values of β, the quasi-static limit ($f \to 0$) provides a viable approximation to the power for values of f less than about one half.

Although the conclusion that the quasi-static limit provides a viable approximation for $f \lesssim \frac{1}{2}$ is based on the results of our analysis of the Witch of Agnesi profile, it is probable that a similar conclusion is valid for more general obstacles. In the general case, we have the asymptotic form

$$E \sim \frac{2}{\pi^2}(1-f^2)^{\frac{1}{2}} \int_0^\infty k\,|\zeta|^2 dk \tag{5.11}$$

in the limit $\beta \to \infty$, provided that $h(x)$ is $n_0 - 1$ times continuously differentiable. It is then apparent that setting $f = 0$ results in at worst a 15 % error in the power

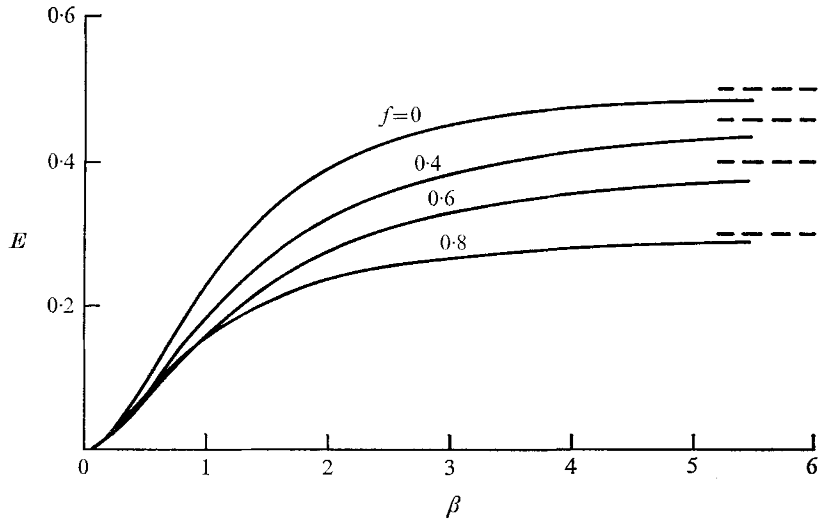

FIGURE 5. Normalized power E_A in internal wave field generated by simple harmonic stratified flow over Witch of Agnesi profile, as a function of β for various values of f. – – – –, limiting values of E_A for $\beta \to \infty$.

estimate in the limit of large β.† By the same token, it may be inferred from the general result for small β, (5.14), that the quasi-static approximation yields a correct result for the power to within a factor of 2 for $f < \frac{1}{2}$ (see also figure 6). These general asymptotic results for large and small β, when combined with the calculated results for the Witch of Agnesi profile for intermediate values of β, tend to support the conclusion that, in general, the quasi-static approximation may be expected to provide a viable estimate of the power for values of f less than about $\frac{1}{2}$.

The behaviour of E_A for $\beta < 1$ is illustrated in figure 6, in which we have plotted E_A/β^2 as a function of β for the values of f included in figure 5. The form of E_A as $\beta \to 0$ is obtained as follows. Referring to (5.8), we may integrate by parts (see Watson 1966, §5.12), to obtain

$$\int_0^\infty k^{-1} \exp(-2k)\, J_n^2\left(\frac{\sqrt{2k}}{\beta f}\right) dk = \frac{1}{2n}\left[1 - 2\int_0^\infty \exp(-2k)\left\{J_0^2\left(\frac{\sqrt{2k}}{\beta f}\right) + 2\sum_{m=1}^{n-1} J_m^2\left(\frac{\sqrt{2k}}{\beta f}\right) + J_n^2\left(\frac{\sqrt{2k}}{\beta f}\right)\right\} d\kappa\right].$$

Referring to Watson (1966, §13.22), we have that

$$\int_0^\infty \exp(-2k)\, J_m^2\left(\frac{\sqrt{2k}}{\beta f}\right) dk = \frac{\beta f}{\sqrt{2\pi}}\, Q_{m-\frac{1}{2}}(1+\beta^2 f^2),$$

where Q is the Legendre function of the second kind. Now, as $\beta^2 f^2 \to 0$,

$$Q_{m-\frac{1}{2}}(1+\beta^2 f^2) \sim -\tfrac{1}{2}\ln\left(\tfrac{1}{2}\beta^2 f^2\right) - \gamma - \psi(m+\tfrac{1}{2})$$

† It is readily verified that the general expression for the power in the steady case (see Miles 1969, (7.3)) reduces to (5.11), with $f = 0$ in the limit $\beta \to \infty$ for continuous $h(x)$.

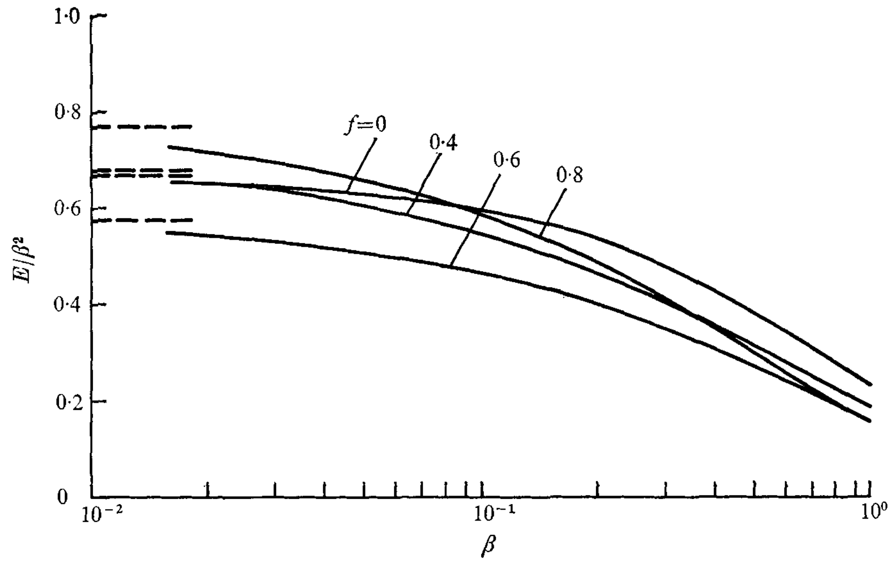

FIGURE 6. Normalized power E_A for $\beta < 1$ and various values of f, plotted as E_A/β^2. – – –, limiting values of E_A/β^2 for $\beta \to 0$.

(see Erdéyli *et al.* 1953, §3.9.2), where γ is Euler's constant and ψ is the psi or digamma function. Thus, as $\beta^2 f^2 \to 0$, we have

$$E_A \sim 2\beta^2 \sum_{n=1}^{n_0} nf^2(1-n^2f^2)^{\frac{1}{2}}, \tag{5.12}$$

although the approach to this asymptotic value is rather slow, the error being of order $\beta f \ln \beta f$. As $f \to 0$, the summation in (5.12) tends to an integral,

$$\int_0^1 \nu(1-\nu^2)^{\frac{1}{2}}\,d\nu = \tfrac{1}{3},$$

so that, for $f = 0$, $E_A \sim \frac{2}{3}\beta^2$, in agreement with Miles & Huppert's result.

As discussed by Miles & Huppert, the limit $\beta \to 0$ is properly identified as the Rayleigh scattering limit. In this limit, the flow field is locally potential in the vicinity of the obstacle, with the effects of stratification being felt only as distances of order U/N from the obstacle. In the Rayleigh scattering limit, then, we may replace the obstacle by an equivalent dipole, so that, in general, we have, as $\beta \to 0$,

$$E \sim \frac{4}{\pi^2}|\zeta|^2_{k=0}\,\beta^2 \sum_{n=1}^{n_0} n^2f^2(1-n^2f^2)^{\frac{1}{2}} \int_0^\infty k^{-1} J_n^2\left(\frac{\sqrt{2k}}{\beta f}\right) dk. \tag{5.13}$$

The integral in (5.13) is simply a special case of the Weber–Schafheitlin discontinuous integral (see Watson 1966, §13.42), so that, in general, we have

$$E \sim \frac{2}{\pi^2}|\zeta|^2_{k=0}\,\beta^2 \sum_{n=1}^{n_0} nf^2(1-n^2f^2)^{\frac{1}{2}} \tag{5.14}$$

in the Rayleigh scattering limit, $\beta \to 0$.

Although much of the discussion in this section concerns a specific example, based on the Witch of Agnesi profile, it is to be expected that the results are of more general interest, especially those on the applicability of the various limiting forms. Of special interest are the conditions under which the bulk of the power in the wave field is concentrated in waves of fundamental frequency. For $\beta \gg 1$, the dominant contribution to the power was found to reside in the waves of fundamental frequency for all but the smallest values of f. But, for $\beta \ll 1$, as indicated by the form of (5.14), this is certainly not the case. There it is immediately apparent that the fundamental waves may be considered dominant only for f somewhat greater than $1/\sqrt{5}$, since below this value the contribution for $n = 2$ is greater than that for $n = 1$. This observation suggests a convenient division of parameter space for the purposes of estimating the flux of energy into the internal wave field. For values of $f = \omega_0/N$ less than about $\frac{1}{2}$, the quasi-static approximation should yield power estimates that are at worst approximately 25 % low. On the other hand, for values in excess of $\frac{1}{2}$, viable estimates of the power may be obtained by considering only the contribution associated with the fundamental frequency ω_0. For β greater than about 2, this contribution may be adequately estimated by the acoustic limit, as expressed by (5.11).

6. Conclusion

We have investigated in some detail the process of internal gravity wave generation by the back-and-forth flow of a stably stratified fluid over an obstacle. To isolate the essential physics of the phenomenon, we have retained only the essential elements of stable density stratification (characterized by a constant Brunt–Väisälä frequency N) and a simple harmonic time dependence in the background flow ($U = U_0 \cos \omega_0 t$). The fundamental solution derived here was found to compare favourably with the limited available experimental evidence. Special attention was given to the behaviour of the solution in several important limiting cases, and to estimating the flux of energy into the internal wave field. Although in general waves are generated not only at the fundamental frequency ω_0, but also at all of its harmonics, it is found that, for values of ω_0/N greater than about one half, the waves of fundamental frequency are dominant. On the other hand, for values of ω_0/N less than about one half, the quasi-static approximation, in which the problem is considered as a slowly-varying version of the classical lee wave problem, is found to provide a viable estimate for the wave field.

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